Topological Dynamics: 
Minimality, Entropy and Chaos.

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4. On Chaotic Interval Maps
Appearance of the Word Chaos in Mathematical Literature in Relation to Maps

Throughout this part of the lecture, where $I$ is a compact interval (usually $I = [0, 1]$) and $f: I \to I$ is a continuous map. The word chaos was first used by Li and Yorke without formal definition. They wrote 'In this work, we analyze the case where the sequence $F^n(x)$ is not periodic and can be called chaotic...'

The main mathematical result of their work contains, in particular, the following statement: Let a continuous map $f \in C(I, I)$, where $I$ is an interval of the real line, have a periodic point of period 3. Then the following statements are true:
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$f$ has a periodic point of any period $k = 1, 2, \ldots$; there exists an uncountable subset $S \subset I$ that does not contain periodic points and satisfies the following conditions:

(a) for any two different points $x \neq y \in S$, one has
$$\lim \inf_{n \to \infty} |f^n(x) - f^n(y)| = 0 \quad \text{and} \quad \lim \sup_{n \to \infty} |f^n(x) - f^n(y)| > 0;$$

(b) for any point $x \in S$ and any periodic point $p \in I$, one has
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It should be noted that Western scientists did not know at that time that the first part of the result of Li and Yorke is a particular case of the general theorem proved by Sharkovsky about 10 years earlier (now known as the Sharkovsky theorem).

Recall the statement of this theorem.

Consider the following ordering (Sharkovsky ordering) in the set $\mathbb{N} \cup \{2^\infty\}$:

$3 \succ 5 \succ 7 \succ \cdots \succ 2 \cdot 3 \succ 2 \cdot 5 \succ 2 \cdot 7 \succ \cdots \succ 4 \cdot 3 \succ 4 \cdot 5 \succ 4 \cdot 7 \succ \cdots \succ 2^n \succ \cdots \succ 4 \succ 2 \succ 1$. 

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Sharkovsky Theorem

For $t \in \mathbb{N} \cup \{2^\infty\}$ let $S(t) = \{k \in \mathbb{N} : t \geq k\}$ (\(S(2^\infty)\) is the set \{1, 2, 4, \ldots, 2^k, \ldots\}), and, for $f \in C(I)$ let $P(f)$ be the set of periods of the periodic points of $f$.

**Sharkovsky theorem**

For any $f \in C(I)$, there exists $t \in \mathbb{N} \cup \{2^\infty\}$ such that $P(f) = S(t)$.

Conversely, for any $t \in \mathbb{N} \cup \{2^\infty\}$, there exists $f \in C(I)$ such that $P(f) = S(t)$.

If $P(f) = S(t)$, then we say that $f$ is of type $t$.

Speaking about types, we assume that the types are well ordered with respect to the Sharkovsky ordering.

A map $f \in C(I)$ is sometimes called topologically chaotic if its type is greater than $2^\infty$ [in this case, this is equivalent to the positivity of the topological entropy of $f$].
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If $P(f) = S(t)$, then we say that $f$ is of type $t$. Speaking about types, we assume that the types are well ordered with respect to the Sharkovsky ordering. A map $f \in C(I)$ is sometimes called *topologically chaotic* if its type is greater than $2^\infty$ (in this case, this is equivalent to the positivity of the topological entropy of $f$).
As mentioned above, the LiYorke definition of chaos for continuous maps of an interval was basically given by Li and Yorke. It is easy to see that if $S$ is uncountable, then there exists at most one point $x \in S$ for which one can find a periodic point $p \in I$ such that $\limsup_{n \to \infty} |f^n(x) - f^n(p)| = 0$. In other words, condition (b) can be omitted, and LiYorke chaos can be regarded as the existence of an uncountable set $S$ with property (a).

It was proved later that all continuous maps of an interval of type greater than $2^\infty$ and some continuous maps of an interval of type $2^\infty$ possess the property of LiYorke chaoticity, and some other equivalent definitions were found. Finally, Kuchta and Smital showed that a map $f \in C(I)$ is LiYorke chaotic if and only if there exist two points $x, y \in I$ such that $\liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0$, but $\limsup_{n \to \infty} |f^n(x) - f^n(y)| > 0$ (in this case, the set $\{x, y\}$ is called a two-point scrambled set).
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General references


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Exercise 4.1

The maps $g : I \rightarrow I$ and $g : I \rightarrow I$, where $I = [0, 1]$, defined by $g(x) = 4x(1 - x)$ and $f(x) = 1 - |2x - 1|$ are topologically chaotic (have positive topological entropy). To prove that the maps $g$ and $f$ are Li-Yorke chaotic.